

SECTION 6-3 Double-Angle and Half-Angle Identities

- Double-Angle Identities
- Half-Angle Identities

This section develops another important set of identities called *double-angle* and *half-angle identities*. We can derive these identities directly from the sum and difference identities given in Section 6-2. Even though the names use the word “angle,” the new identities hold for real numbers as well.

• Double-Angle Identities

Start with the sum identity for sine,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

and replace y with x to obtain

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$

On simplification, this gives

$$\sin 2x = 2 \sin x \cos x \quad \text{Double-angle identity for sine} \quad (1)$$

If we start with the sum identity for cosine,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

and replace y with x , we obtain

$$\cos(x + x) = \cos x \cos x - \sin x \sin x$$

On simplification, this gives

$$\cos 2x = \cos^2 x - \sin^2 x \quad \text{First double-angle identity for cosine} \quad (2)$$

Now, using the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1 \quad (3)$$

in the form

$$\cos^2 x = 1 - \sin^2 x \quad (4)$$

and substituting it into equation (2), we get

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

On simplification, this gives

$$\cos 2x = 1 - 2 \sin^2 x \quad \text{Second double-angle identity for cosine} \quad (5)$$

Or, if we use equation (3) in the form

$$\sin^2 x = 1 - \cos^2 x$$

and substitute it into equation (2), we get

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$

On simplification, this gives

$$\cos 2x = 2 \cos^2 x - 1 \quad \text{Third double-angle identity for cosine} \quad (6)$$

Double-angle identities can be established for the tangent function in the same way by starting with the sum formula for tangent (a good exercise for you).

We list the double-angle identities below for convenient reference.

Double-Angle Identities

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} = \frac{2 \cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

The identities in the second row are used to a good advantage in calculus in the form

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to transform a power form to a nonpower form.

EXPLORE-DISCUSS 1 (A) Discuss how you would show that, in general,

$$\sin 2x \neq 2 \sin x \quad \cos 2x \neq 2 \cos x \quad \tan 2x \neq 2 \tan x$$



(B) Graph $y_1 = \sin 2x$ and $y_2 = 2 \sin x$ in the same viewing window. Conclusion? Repeat the process for the other two statements in part A.

EXAMPLE 1 Identity Verification

Verify the identity: $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

Verification We start with the right side:

$$\begin{aligned} \frac{1 - \tan^2 x}{1 + \tan^2 x} &= \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\sin^2 x}{\cos^2 x}} && \text{Quotient identities} \\ &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} && \text{Algebra} \\ &= \cos^2 x - \sin^2 x && \text{Pythagorean Identity} \\ &= \cos 2x && \text{Double-angle identity} \end{aligned}$$

Key Algebraic Steps in Example 1

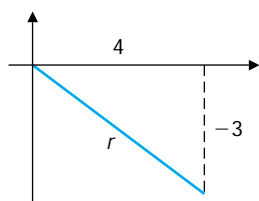
$$\frac{1 - \frac{a^2}{b^2}}{1 + \frac{a^2}{b^2}} = \frac{b^2 \left(1 - \frac{a^2}{b^2}\right)}{b^2 \left(1 + \frac{a^2}{b^2}\right)} = \frac{b^2 - a^2}{b^2 + a^2}$$

Matched Problem 1 Verify the identity: $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$

EXAMPLE 2 Finding Exact Values

Find the exact values, without using a calculator, of $\sin 2x$ and $\cos 2x$ if $\tan x = -\frac{3}{4}$ and x is a quadrant IV angle.

Solution First draw the reference triangle for x and find any unknown sides:



$$r = \sqrt{(-3)^2 + 4^2} = 5$$

$$\sin x = -\frac{3}{5}$$

$$\cos x = \frac{4}{5}$$

Now use double-angle identities for sine and cosine:

$$\sin 2x = 2 \sin x \cos x = 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos 2x = 2 \cos^2 x - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{7}{25}$$

Matched Problem 2 Find the exact values, without using a calculator, of $\cos 2x$ and $\tan 2x$ if $\sin x = \frac{4}{5}$ and x is a quadrant II angle.

• **Half-Angle Identities**

Half-angle identities are simply double-angle identities stated in an alternate form. Let's start with the double-angle identity for cosine in the form

$$\cos 2m = 1 - 2 \sin^2 m$$

Now replace m with $x/2$ and solve for $\sin(x/2)$ [if $2m$ is twice m , then m is half of $2m$ —think about this]:

$$\cos x = 1 - 2 \sin^2 \frac{x}{2}$$

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \quad \text{Half-angle identity for sine} \quad (7)$$

where the choice of the sign is determined by the quadrant in which $x/2$ lies.

To obtain a half-angle identity for cosine, start with the double-angle identity for cosine in the form

$$\cos 2m = 2 \cos^2 m - 1$$

and let $m = x/2$ to obtain

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}} \quad \text{Half-angle identity for cosine} \quad (8)$$

where the sign is determined by the quadrant in which $x/2$ lies.

To obtain a *half-angle identity for tangent*, use the quotient identity and the half-angle formulas for sine and cosine:

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\pm \sqrt{\frac{1 - \cos x}{2}}}{\pm \sqrt{\frac{1 + \cos x}{2}}} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

Thus,

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad \text{Half-angle identity for tangent} \quad (9)$$

where the sign is determined by the quadrant in which $x/2$ lies.

Simpler versions of equation (9) can be obtained as follows:

$$\begin{aligned} \left| \tan \frac{x}{2} \right| &= \sqrt{\frac{1 - \cos x}{1 + \cos x}} \\ &= \sqrt{\frac{1 - \cos x}{1 + \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}} \end{aligned} \quad (10)$$

$$\begin{aligned}
 &= \sqrt{\frac{1 - \cos^2 x}{(1 + \cos x)^2}} \\
 &= \sqrt{\frac{\sin^2 x}{(1 + \cos x)^2}} \\
 &= \frac{\sqrt{\sin^2 x}}{\sqrt{(1 + \cos x)^2}} \\
 &= \frac{|\sin x|}{1 + \cos x} \qquad \sqrt{\sin^2 x} = |\sin x| \text{ and} \\
 &\qquad \qquad \qquad \sqrt{(1 + \cos x)^2} = 1 + \cos x, \text{ since } 1 + \cos x \text{ is never negative.}
 \end{aligned}$$

All absolute value signs can be dropped, since it can be shown that $\tan(x/2)$ and $\sin x$ always have the same sign (a good exercise for you). Thus,

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} \quad \text{Half-angle identity for tangent} \quad (11)$$

By multiplying the numerator and the denominator in the radicand in equation (10) by $1 - \cos x$ and reasoning as before, we also can obtain

$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} \quad \text{Half-angle identity for tangent} \quad (12)$$

We now list all the half-angle identities for convenient reference.

Half-Angle Identities

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

where the sign is determined by the quadrant in which $x/2$ lies.

EXPLORE-DISCUSS 2 (A) Discuss how you would show that, in general,

$$\sin \frac{x}{2} \neq \frac{1}{2} \sin x \qquad \cos \frac{x}{2} \neq \frac{1}{2} \cos x \qquad \tan \frac{x}{2} \neq \frac{1}{2} \tan x$$



(B) Graph $y_1 = \sin(x/2)$ and $y_2 = \frac{1}{2} \sin x$ in the same viewing window. Conclusion? Repeat the process for the other two statements in part A.

EXAMPLE 3 Finding Exact Values

Compute the exact value of $\sin 165^\circ$ without a calculator using a half-angle identity.

Solution

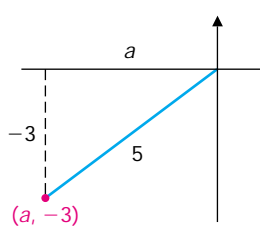
$$\begin{aligned}\sin 165^\circ &= \sin \frac{330^\circ}{2} \\ &= \sqrt{\frac{1 - \cos 330^\circ}{2}} && \text{Use half-angle identity for sine with a positive radical, since } \sin 165^\circ \text{ is positive.} \\ &= \sqrt{\frac{1 - (\sqrt{3}/2)}{2}} \\ &= \frac{\sqrt{2 - \sqrt{3}}}{2}\end{aligned}$$

Matched Problem 3 Compute the exact value of $\tan 105^\circ$ without a calculator using a half-angle identity.

EXAMPLE 4 Finding Exact Values

Find the exact values of $\cos (x/2)$ and $\cot (x/2)$ without using a calculator if $\sin x = -\frac{3}{5}$, $\pi < x < 3\pi/2$.

Solution Draw a reference triangle in the third quadrant, and find $\cos x$. Then use appropriate half-angle identities.



$$\begin{aligned}a &= -\sqrt{5^2 - (-3)^2} = -4 \\ \cos x &= -\frac{4}{5}\end{aligned}$$

If $\pi < x < 3\pi/2$, then

$$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \quad \text{Divide each member of } \pi < x < 3\pi/2 \text{ by 2.}$$

Thus, $x/2$ is an angle in the second quadrant where cosine and cotangent are negative, and

$$\begin{aligned}\cos \frac{x}{2} &= -\sqrt{\frac{1 + \cos x}{2}} && \cot \frac{x}{2} = \frac{1}{\tan (x/2)} = \frac{\sin x}{1 - \cos x} \\ &= -\sqrt{\frac{1 + (-\frac{4}{5})}{2}} && = \frac{-\frac{3}{5}}{1 - (-\frac{4}{5})} = -\frac{1}{3} \\ &= -\sqrt{\frac{1}{10}} \text{ or } \frac{-\sqrt{10}}{10}\end{aligned}$$

Matched Problem 4 Find the exact values of $\sin(x/2)$ and $\tan(x/2)$ without using a calculator if $\cot x = -\frac{4}{3}$, $\pi/2 < x < \pi$.

EXAMPLE 5 Identity Verification

Verify the identity: $\sin^2 \frac{x}{2} = \frac{\tan x - \sin x}{2 \tan x}$

Verification $\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$ Half-angle identity for sine

$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$ Square both sides.

$= \frac{\tan x}{\tan x} \cdot \frac{1 - \cos x}{2}$ Algebra

$= \frac{\tan x - \tan x \cos x}{2 \tan x}$ Algebra

$= \frac{\tan x - \sin x}{2 \tan x}$ Quotient identity

Matched Problem 5 Verify the identity: $\cos^2 \frac{x}{2} = \frac{\tan x + \sin x}{2 \tan x}$

Answers to Matched Problems

$$1. \frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \left(\frac{\sin x}{\cos x} \right)}{1 + \frac{\sin^2 x}{\cos^2 x}} = \frac{\cos^2 x \left[2 \left(\frac{\sin x}{\cos x} \right) \right]}{\cos^2 x \left(1 + \frac{\sin^2 x}{\cos^2 x} \right)} = \frac{2 \sin x \cos x}{\cos^2 x + \sin^2 x} = 2 \sin x \cos x = \sin 2x$$

$$2. \cos 2x = -\frac{7}{25}, \tan 2x = \frac{24}{7} \quad 3. -\sqrt{3} - 2 \quad 4. \sin(x/2) = 3\sqrt{10}/10, \tan(x/2) = 3$$

$$5. \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} = \frac{\tan x}{\tan x} \cdot \frac{1 + \cos x}{2} = \frac{\tan x + \tan x \cos x}{2 \tan x} = \frac{\tan x + \sin x}{2 \tan x}$$

EXERCISE 6-3

A

In Problems 1–6, verify each identity for the values indicated.

1. $\cos 2x = \cos^2 x - \sin^2 x$, $x = 30^\circ$

2. $\sin 2x = 2 \sin x \cos x$, $x = 45^\circ$

3. $\tan 2x = \frac{2}{\cot x - \tan x}$, $x = \frac{\pi}{3}$


4. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$, $x = \frac{\pi}{6}$

5. $\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$, $x = \pi$
(Choose the correct sign.)

6. $\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$, $x = \frac{\pi}{2}$
(Choose the correct sign.)

In Problems 7–10, find the exact value without a calculator using half-angle identities.

7. $\tan 15^\circ$ 8. $\sin 165^\circ$
 9. $\cos 112.5^\circ$ 10. $\tan 157.5^\circ$

 In Problems 11–14, graph y_1 and y_2 in the same viewing window for $-2\pi \leq x \leq 2\pi$. Use TRACE to compare the two graphs.

11. $y_1 = \cos 2x$, $y_2 = \cos^2 x - \sin^2 x$
 12. $y_1 = \sin 2x$, $y_2 = 2 \sin x \cos x$
 13. $y_1 = \tan \frac{x}{2}$, $y_2 = \frac{\sin x}{1 + \cos x}$
 14. $y_1 = \tan 2x$, $y_2 = \frac{2 \tan x}{1 - \tan^2 x}$

B

Verify the identities in Problems 15–28.

15. $(\sin x + \cos x)^2 = 1 + \sin 2x$
 16. $\sin 2x = (\tan x)(1 + \cos 2x)$
 17. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
 18. $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$
 19. $1 - \cos 2x = \tan x \sin 2x$
 20. $1 + \sin 2t = (\sin t + \cos t)^2$
 21. $\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$ 22. $\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$
 23. $\cot \frac{\theta}{2} = \frac{\sin \theta}{1 - \cos \theta}$ 24. $\cot \frac{\theta}{2} = \frac{1 + \cos \theta}{\sin \theta}$
 25. $\cos 2u = \frac{1 - \tan^2 u}{1 + \tan^2 u}$ 26. $\frac{\cos 2u}{1 - \sin 2u} = \frac{1 + \tan u}{1 - \tan u}$
 27. $2 \csc 2x = \frac{1 + \tan^2 x}{\tan x}$ 28. $\sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}$

In Problems 29–34, is the equation an identity? Explain.

29. $\cos 2x = 2 \sin x \cos x$ 30. $\sin 4x = 4 \sin x \cos x$
 31. $\tan 2x = \frac{-2 \tan x}{\tan^2 x - 1}$ 32. $\tan 6x = \frac{6 \tan x}{1 - \tan^2 x}$
 33. $\cot 2x = \frac{2 \cot x}{1 - \cot^2 x}$ 34. $2 \csc 2x = \sec x \csc x$

Compute the exact values of $\sin 2x$, $\cos 2x$, and $\tan 2x$ using the information given in Problems 35–38 and appropriate identities. Do not use a calculator.

35. $\sin x = \frac{3}{5}$, $\pi/2 < x < \pi$ 36. $\cos x = -\frac{4}{5}$, $\pi/2 < x < \pi$
 37. $\tan x = -\frac{5}{12}$, $-\pi/2 < x < 0$
 38. $\cot x = -\frac{5}{12}$, $-\pi/2 < x < 0$

In Problems 39–42, compute the exact values of $\sin(x/2)$, $\cos(x/2)$, and $\tan(x/2)$ using the information given and appropriate identities. Do not use a calculator.


39. $\sin x = -\frac{1}{3}$, $\pi < x < 3\pi/2$
 40. $\cos x = -\frac{1}{4}$, $\pi < x < 3\pi/2$
 41. $\cot x = \frac{3}{4}$, $-\pi < x < -\pi/2$
 42. $\tan x = \frac{3}{4}$, $-\pi < x < -\pi/2$

Suppose you are tutoring a student who is having difficulties in finding the exact values of $\sin \theta$ and $\cos \theta$ from the information given in Problems 43 and 44. Assuming that you have worked through each problem and have identified the key steps in the solution process, proceed with your tutoring by guiding the student through the solution process using the following questions. Record the expected correct responses from the student.

- (A) The angle 2θ is in what quadrant and how do you know?
 (B) How can you find $\sin 2\theta$ and $\cos 2\theta$? Find each.
 (C) What identities relate $\sin \theta$ and $\cos \theta$ with either $\sin 2\theta$ or $\cos 2\theta$?
 (D) How would you use the identities in part C to find $\sin \theta$ and $\cos \theta$ exactly, including the correct sign?
 (E) What are the exact values for $\sin \theta$ and $\cos \theta$?
 43. Find the exact values of $\sin \theta$ and $\cos \theta$, given $\tan 2\theta = -\frac{4}{3}$, $0^\circ < \theta < 90^\circ$.
 44. Find the exact values of $\sin \theta$ and $\cos \theta$, given $\sec 2\theta = -\frac{5}{4}$, $0^\circ < \theta < 90^\circ$.

Verify each of the following identities for the value of x indicated in Problems 45–48. Compute values to 5 significant digits using a calculator.

- (A) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$
 (B) $\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$
 (Choose the correct sign.)
 45. $x = 252.06^\circ$ 46. $x = 72.358^\circ$
 47. $x = 0.93457$ 48. $x = 4$

 In Problems 49–52, graph y_1 and y_2 in the same viewing window for $-2\pi \leq x \leq 2\pi$, and state the intervals for which the equation $y_1 = y_2$ is an identity.

49. $y_1 = \cos(x/2)$, $y_2 = \sqrt{\frac{1 + \cos x}{2}}$
 50. $y_1 = \cos(x/2)$, $y_2 = -\sqrt{\frac{1 + \cos x}{2}}$
 51. $y_1 = \sin(x/2)$, $y_2 = -\sqrt{\frac{1 - \cos x}{2}}$
 52. $y_1 = \sin(x/2)$, $y_2 = \sqrt{\frac{1 - \cos x}{2}}$

C

Verify the identities in Problems 53–56.

53. $\cos 3x = 4 \cos^3 x - 3 \cos x$

54. $\sin 3x = 3 \sin x - 4 \sin^3 x$

55. $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$

56. $\sin 4x = (\cos x)(4 \sin x - 8 \sin^3 x)$

In Problems 57–62, find the exact value of each without using a calculator:

57. $\cos(2 \cos^{-1} \frac{3}{5})$


58. $\sin(2 \cos^{-1} \frac{3}{5})$

59. $\tan[2 \cos^{-1}(-\frac{4}{5})]$

60. $\tan[2 \tan^{-1}(-\frac{3}{4})]$

61. $\cos[\frac{1}{2} \cos^{-1}(-\frac{3}{5})]$

62. $\sin[\frac{1}{2} \tan^{-1}(-\frac{4}{3})]$

 In Problems 63–68, graph $f(x)$ in a graphing utility, find a simpler function $g(x)$ that has the same graph as $f(x)$, and verify the identity $f(x) = g(x)$. [Assume $g(x) = k + AT(Bx)$, where $T(x)$ is one of the six trigonometric functions.]

63. $f(x) = \csc x - \cot x$

64. $f(x) = \csc x + \cot x$

65. $f(x) = \frac{1 - 2 \cos 2x}{2 \sin x - 1}$

66. $f(x) = \frac{1 + 2 \cos 2x}{1 + 2 \cos x}$

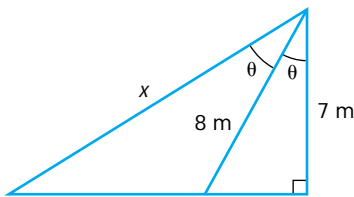
67. $f(x) = \frac{1}{\cot x \sin 2x - 1}$

68. $f(x) = \frac{\cot x}{1 + \cos 2x}$

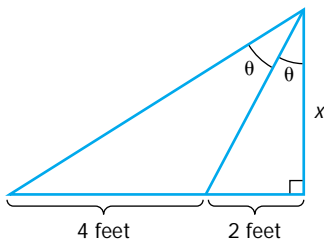
APPLICATIONS



- ★ 69. **Indirect Measurement.** Find the exact value of x in the figure; then find x and θ to three decimal places. [Hint: Use $\cos 2\theta = 2 \cos^2 \theta - 1$.]



- ★ 70. **Indirect Measurement.** Find the exact value of x in the figure; then find x and θ to three decimal places. [Hint: Use $\tan 2\theta = (2 \tan \theta)/(1 - \tan^2 \theta)$.]

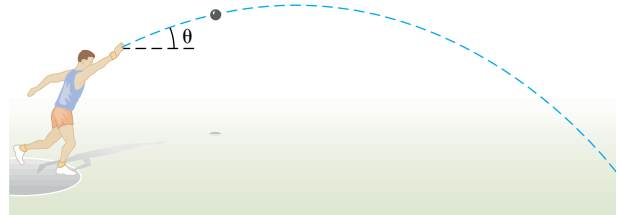


- ★ 71. **Sports—Physics.** The theoretical distance d that a shot-putter, discus thrower, or javelin thrower can achieve on a

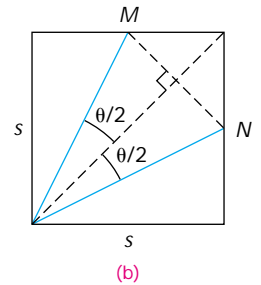
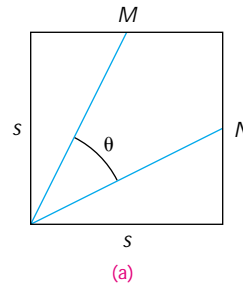
given throw is found in physics to be given approximately by

$$d = \frac{2v_0^2 \sin \theta \cos \theta}{32 \text{ feet per second per second}}$$

where v_0 is the initial speed of the object thrown (in feet per second) and θ is the angle above the horizontal at which the object leaves the hand (see the figure).



- (A) Write the formula in terms of $\sin 2\theta$ by using a suitable identity.
 (B) Using the resulting equation in part A, determine the angle θ that will produce the maximum distance d for a given initial speed v_0 . This result is an important consideration for shot-putters, javelin throwers, and discus throwers.
72. **Geometry.** In part (a) of the figure, M and N are the midpoints of the sides of a square. Find the exact value of $\cos \theta$. [Hint: The solution uses the Pythagorean theorem, the definition of sine and cosine, a half-angle identity, and some auxiliary lines as drawn in part (b) of the figure.]



-  73. **Area.** An n -sided regular polygon is inscribed in a circle of radius R .

(A) Show that the area of the n -sided polygon is given by

$$A_n = \frac{1}{2}nR^2 \sin \frac{2\pi}{n}$$

[Hint: Area of a triangle = $\frac{1}{2}$ (base)(altitude). Also, a double-angle identity is useful.]

(B) For a circle of radius 1, complete Table 1, to five decimal places, using the formula in part A.

TABLE 1

n	10	100	1,000	10,000
A_n				

- (C) What number does A_n seem to approach as n increases without bound? (What is the area of a circle of radius 1?)
- (D) Will A_n exactly equal the area of the circumscribed circle for some sufficiently large n ? How close can A_n be made to get to the area of the circumscribed circle. [In

calculus, the area of the circumscribed circle is called the *limit* of A_n as n increases without bound. In symbols, for a circle of radius 1, we would write $\lim_{n \rightarrow \infty} A_n = \pi$. The limit concept is the cornerstone on which calculus is constructed.]

SECTION 6-4 Product–Sum and Sum–Product Identities

- Product–Sum Identities
- Sum–Product Identities

Our work with identities is concluded by developing the *product–sum* and *sum–product identities*, which are easily derived from the sum and difference identities developed in Section 6-2. These identities are used in calculus to convert product forms to more convenient sum forms. They also are used in the study of sound waves in music to convert sum forms to more convenient product forms.

- **Product–Sum Identities**

First, add, left side to left side and right side to right side, the sum and difference identities for sine:

$$\begin{array}{r} \sin(x + y) = \sin x \cos y + \cos x \sin y \\ \sin(x - y) = \sin x \cos y - \cos x \sin y \\ \hline \sin(x + y) + \sin(x - y) = 2 \sin x \cos y \end{array}$$

or

$$\sin x \cos y = \frac{1}{2} [\sin(x + y) + \sin(x - y)]$$

Similarly, by adding or subtracting the appropriate sum and difference identities, we can obtain three other **product–sum identities**. These are listed below for convenient reference.

Product–Sum Identities

$$\begin{aligned} \sin x \cos y &= \frac{1}{2} [\sin(x + y) + \sin(x - y)] \\ \cos x \sin y &= \frac{1}{2} [\sin(x + y) - \sin(x - y)] \\ \sin x \sin y &= \frac{1}{2} [\cos(x - y) - \cos(x + y)] \\ \cos x \cos y &= \frac{1}{2} [\cos(x + y) + \cos(x - y)] \end{aligned}$$

EXAMPLE 1 A Product as a Difference

Write the product $\cos 3t \sin t$ as a sum or difference.